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# BODY WITH MIRROR SURFACE AND CONNECTED INTERIOR INVISIBLE FROM ONE POINT

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**Abstract** Here we demonstrate existence of a piecewise smooth obstacle having connected interior and invisible from a point in the framework of geometric optics.

**Key words:** Invisibility, billiards, geometrical optics.

**AMS Mathematics Subject Classification:** 37D50, 49Q10

## 1 Introduction

The scattering theory prohibits existence of absolutely invisible bodies, since a nontrivial outgoing solution of the Helmholtz equation cannot have zero scattering amplitude. Nevertheless, invisibility is possible in the framework of geometric optics which involves mathematical design of bodies with well-defined surfaces whose scattering map preserves certain trajectories of a flow of elastic particles. The main practical application of this study is optical shielding: by surrounding an object by a specially designed mirror surface, it is possible to create an illusion of invisibility from given points or directions.

The first work that targets the problem of designing a body invisible in a direction in the framework of mirror invisibility appears in [1] and is motivated by the problem of constructing a nonconvex body or zero resistance. The authors demonstrated that there exists a (connected and even simply connected) body invisible in one direction: if this body is manufactured out of perfectly reflective mirrors, a laser beam sent through this construction in the direction of invisibility would leave the body along the same trajectory. Remarkably, in [5], [6], [7] the scattering of acoustic waves by this body was studied.

This pioneering research led to several intriguing mathematical problems. One of them, proposed by Sergei Tabachnikov [4], asks whether it is possible to design a body with mirror surface invisible in two directions. The problem was solved by Plakhov and Roshchina in [8]: it was shown that a construction combining several pieces of parabolic cylinders can be used to produce a body invisible in two directions in the three-dimensional case. This body consists of two connected components, and its interior consists of 8 connected components, so it looks complicated to use such a construction in practical applications. The main result of the paper is the following Theorem 1 (see figure 2).

**Theorem 1.1.** *Given a point in $\mathbb{R}^3$, there exists a body in $\mathbb{R}^3$ with connected interior which is invisible from this point.*

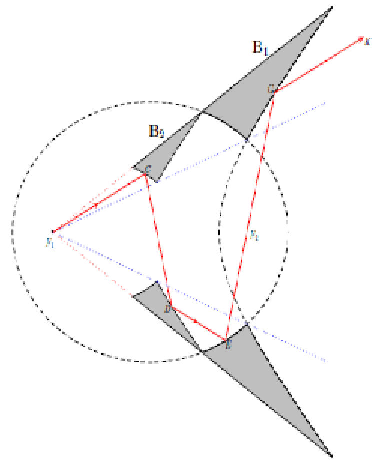


Figure 1: A body invisible from one point.

## 1 Definitions

We begin with reminding relevant definitions, then explain our construction and prove that it is invisible from two points.

**Definition 1.** A *body* is a finite or countable union of its connected components, where each component is an open bounded domain with piecewise smooth boundary.

**Definition 2.** A body $B \subset \mathbb{R}^d$ is said to be *invisible from a point* $O \in \mathbb{R}^d \setminus B$, if for almost all $v \in S^{d-1}$ the billiard particle in $\mathbb{R}^d \setminus B$ emanating from $O$ with the initial velocity $v$, after a finite number of reflections from $\partial B$ will eventually move freely with the same velocity $v$ along a straight line containing $O$.

If the point $O$ is infinitely distant, we get the notion of a body invisible in a direction.

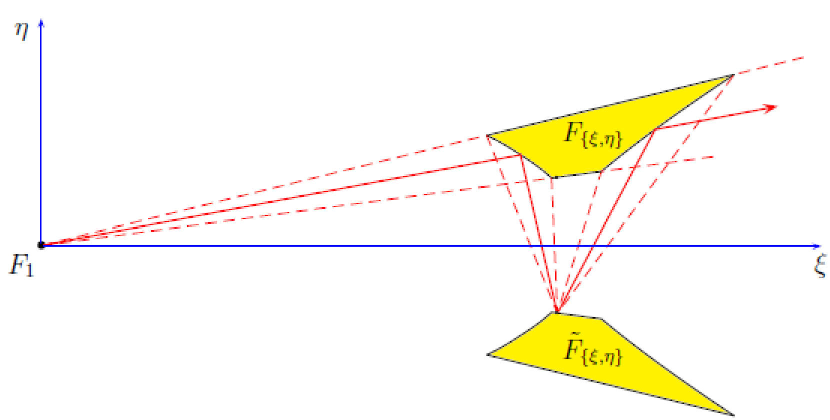


Figure 2: A two-dimensional figure invisible from the origin. The 3-dimensional construction is obtained by rotating this figure around the $\xi$-axis.

Notice that a 3D body invisible from one point was constructed in [8] (a central cross section of this body by a plane passing through the point is shown in Fig. 2). Its interior is disconnected: it consists two connected components. This provides a difficulty in practical realization of this construction. On the contrary, below we construct a body with connected interior.

## 1 Construction

We describe the geometrical shape of the body invisible from one point, provide a proof of its invisibility, and then give exact formulas that determine its shape. The description is made in several steps.

**1.** Consider the ellipse $\mathcal{E}$ given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a > 0, \ b > 0$$

in Cartesian coordinates $x$, $y$. The foci of $\mathcal{E}$ are the points $F_1 = (-c, 0)$ and $F_2 = (c, 0)$, where $c = \sqrt{a^2 - b^2}$. Next consider the hyperbola

$$\frac{x^2}{\alpha^2} - \frac{y^2}{\beta^2} = 1, \quad \alpha > 0, \ \beta > 0.$$

We require that the hyperbola has the same foci $F_1$ and $F_2$, that is, the parameters $\alpha$ and $\beta$ satisfy the equality

$$c = \sqrt{\alpha^2 + \beta^2}. \tag{1}$$

Denote by $\mathcal{H}$ the right branch of the hyperbola. There are two points of intersection of $\mathcal{H}$ with the ellipse $\mathcal{E}$, which are symmetrical to each other with respect to the $x$-axis; we denote by $C$ the upper point of intersection (see Fig. 3). Let us additionally impose

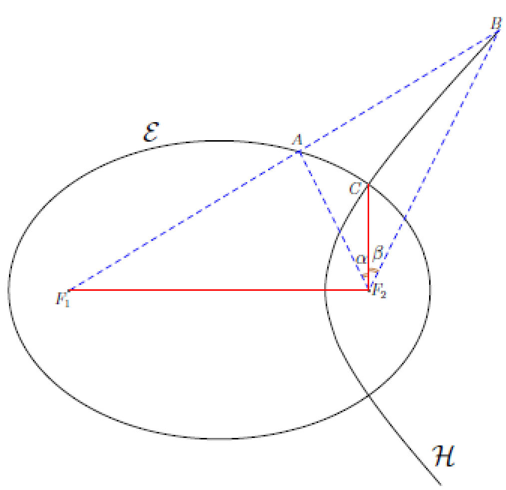


Figure 3: Ellipse and hyperbola.

the condition that the segments $F_1F_2$ and $F_2C$ are perpendicular; one easily sees that this condition is equivalent to the equation

$$\alpha a = c^2.$$

It is convenient to introduce the parameter

$$\varkappa = \frac{a}{c} = \frac{c}{\alpha}. \tag{2}$$

**2.** Here we prove some auxiliary geometric statements which will be needed later on. First state a characteristic property of angle bisector in a triangle.

**Property.** *The segment $f$ is the bisector of the corresponding angle in Figure 4 (that is, $\alpha = \beta$), if and only if $(a_1 + b_1)(a_2 - b_2) = f^2$.*

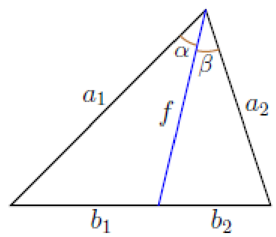


Figure 4: The characteristic property of the angle bisector.

*Sketch of the proof.* Consider the following relations on the values $a_1$, $a_2$, $b_1$, $b_2$, and $f$:

1. $a_1/a_2 = b_1/b_2$;
2. $a_1a_2 - b_1b_2 = f^2$;
3. $(a_1 + b_1)(a_2 - b_2) = f^2$. (3)

The equalities 1 and 2 are well known in the literature; each of them is a characteristic property of triangle bisector. The equality 3 is a direct consequence of the equalities 1 and 2; thus the direct property (3) of the angle bisector is established. The proof of the inverse property (3) is also simple, but cumbersome, and utilizes the sine rule and some trigonometry. It is omitted here. □

**Proposition.** *The angles $\alpha = \measuredangle AF_2C$ and $\beta = \measuredangle BF_2C$ in Figure 3 are equal.*

*Proof.* Let us make an auxiliary construction. Extend the segment $BF_2$ until the second intersection with the ellipse at a point $A'$. Denote by $C'$ the second point of intersection of the ellipse with the branch of the hyperbola $\mathcal{H}$. Denote

$$f = 2c = |F_1F_2|, \;\; g = |F_2C| = |F_2C'|, \;\; a_1 = |F_1A'|,$$

$$b_1 = |F_2A'|, \;\; a_2 = |F_1B|, \text{ and } b_2 = |F_2B|$$

(see Fig. 5). By the focal property of the ellipse, we have $|F_1A'| + |F_2A'| = |F_1C'| + |F_2C'|$, that is,

$$a_1 + b_1 = \sqrt{f^2 + g^2} + g. \tag{4}$$

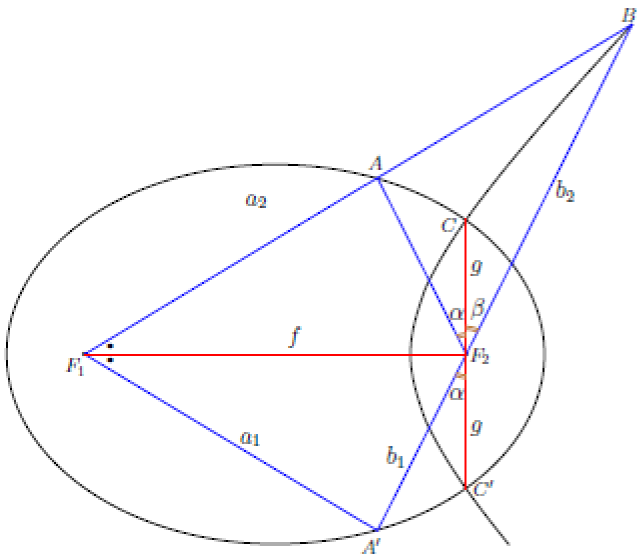


Figure 5: Auxiliary construction.

Further, by the focal property of the hyperbola we have $|F_1B|-|F_2B| = |F_1C|-|F_2C|$, that is,

$$a_2 - b_2 = \sqrt{f^2+g^2} - g. \tag{5}$$

Multiplying both sides of (4) and (5), we get

$$(a_1+b_1)(a_2-b_2) = f^2,$$

and taking into account the Property, one concludes that $F_1F_2$ is the bisector of the angle $F_1$ in the triangle $A'F_1B$. This means that $A'$ is symmetric to $A$ with respect to the straight line $F_1F_2$, and by symmetry one has

$$\measuredangle AF_2C = \measuredangle A'F_2C'. \tag{6}$$

On the other hand, the angles $\measuredangle BF_2C$ and $\measuredangle A'F_2C'$ are vertical, and therefore, are equal:

$$\measuredangle BF_2C = \measuredangle A'F_2C'. \tag{7}$$

The equations (6) and (7) imply that $\measuredangle AF_2C = \measuredangle BF_2C$, therefore $\alpha = \beta$. □

**3.** Draw a ray with the vertex at $F_1$,

$$y = k(x+c), \quad x \geq -c,$$

with $k > 0$. The ray intersects the branch $\mathcal{H}$ of the hyperbola, if and only if $k < \beta/\alpha$. Taking into account the relations (1) and (2) on $\alpha$ and $\beta$, one rewrites this inequality as $k < k_{\max}$, where

$$k_{\max} = \sqrt{\varkappa^2 - 1}. \tag{8}$$

Suppose that $k$ satisfies (8) and denote by $A$ and $B$ the points of intersection of the ray with $\mathcal{E}$ and $\mathcal{H}$, respectively (see Fig. 3).

In what follows we will also assume that the inequalities

$$|F_1A| < |F_1F_2| < |F_1B| \tag{9}$$

are satisfied. Below we derive the condition on $k$ equivalent to (9). Denote $A = (x_A, y_A)$ and $B = (x_B, y_B)$; the following relations can be easily derived:

$$|F_1A| = \frac{c}{a}\,x_A + a \quad \text{and} \quad |F_1B| = \frac{c}{\alpha}\,x_B + \alpha. \tag{10}$$

By the second formula in (10), one has $|F_1B| > |F_1C| > |F_1F_2|$, and so, the second inequality in (9) is always satisfied.

Note that

$$|F_1F_2| = 2c. \tag{11}$$

The ray with the largest inclination $y = k_{\max}(x+c)$ intersects $\mathcal{E}$ at the point $A_\infty = (0, b)$, therefore $|F_1A_\infty| = \sqrt{c^2 + b^2} = a$. We impose the condition

$$\varkappa < 2;$$

then the distance $|F_1A|$ monotonically decreases from $|F_1C| = \sqrt{(2c)^2 + b^4/a^2} > 2c$ to $|F_1A_\infty| = a < 2c$ when $A$ runs the elliptic curve $CA_\infty$ from $C$ to $A_\infty$, and takes the value $2c$ at a single point $A_0$ in between.

Using (11) and the first formula in (10), we conclude that the first inequality in (9) is equivalent to $(c/a)x_A + a < 2c$, which can be rewritten as

$$x_A < x_0 = a\left(2 - \frac{a}{c}\right).$$

Let $A_0 = (x_0, y_0)$ be the point on the ellipse; then one has

$$y_0 = c\sqrt{\varkappa^2 - 1}\sqrt{1 - (2 - \varkappa)^2}.$$

We conclude that the first inequality in (9) is equivalent to $k > k_{\min}$, where

$$k_{\min} = \frac{y_0}{x_0 + c} = \frac{\sqrt{\varkappa^2 - 1}\sqrt{1 - (2 - \varkappa)^2}}{1 + 2\varkappa - \varkappa^2} = (\varkappa - 1)\frac{\sqrt{4 - (\varkappa - 1)^2}}{2 - (\varkappa - 1)^2}. \tag{12}$$

Thus, the condition ensuring that the ray $y = k(x + c)$, $x \ge -c$ intersects both $\mathcal{E}$ and $\mathcal{H}$ and that for the points of intersection, $A$ and $B$, the inequalities (9) are satisfied, reads as

$$k_{\min} < k < k_{\max}.$$

**4.** Draw two rays with inclinations $k_1$ and $k_2$, $y = k_1(x + c)$, $x \ge -c$ and $y = k_2(x + c)$, $x \ge -c$, where

$$k_{\min} < k_1 < k_2 < k_{\max}. \tag{13}$$

The ray $y = k_1(x + c)$, $x \geq -c$ is denoted by $F_1K$ in Figure 6. From the previous item we know that both rays intersect $\mathcal{E}$ and $\mathcal{H}$ and the inequalities (9) are satisfied, with $A$ and $B$ being the points of intersection of $F_1K$ with $\mathcal{E}$ and $\mathcal{H}$.

Determine the figure $F_{\{x,y\}}$ by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} > 1, \quad \frac{x^2}{\alpha^2} - \frac{y^2}{\beta^2} < 1,$$

$$k_1 < \frac{y}{x+c} < k_2, \quad y > 0$$

(see Fig. 6).

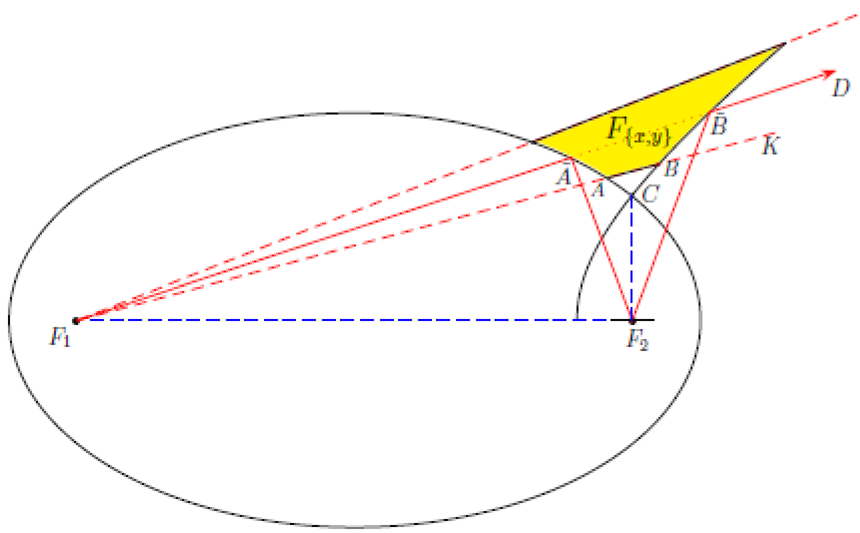


Figure 6: A light ray reflecting from the mirrors.

Take a ray $F_1D$ at an inclination $k \in (k_1, k_2)$. Let $\tilde{A}$ and $\tilde{B}$ be the points of intersection of this ray with the elliptic and hyperbolic arcs forming the boundary of $F_{\{x,y\}}$. Now imagine that the boundary of $F_{\{x,y\}}$ is mirror-like and there is a flat mirror on the line $F_1F_2$. Then the broken line $F_1\tilde{A}F_2\tilde{B}D$ represents a light ray emanating from $F_1$ and making reflections from these mirror boundaries.

Indeed, according to the focal property of the billiard in ellipse, the light ray from $F_1$, after a reflection at $\tilde{A}$, gets into $F_2$. The segment $F_2C$ is orthogonal to $F_1F_2$ and is the bisector of the angle $\tilde{A}F_2\tilde{B}$, as proved in the Proposition. Therefore the light ray, after the second reflection at $F_2$, gets into $\tilde{B}$. According to the focal property of the billiard in hyperbola, the light ray reflected at $\tilde{B}$ moves along the straight line $\tilde{B}D$ through $F_1$.

Now take the angle $\gamma = \frac{1}{2}\arctan k_1 = \frac{1}{2}\measuredangle KF_1F_2$. The tangent $t = \tan\gamma$ satisfies the equation

$$\frac{2t}{1-t^2} = k_1, \tag{14}$$

which implies that
$$t = \frac{\sqrt{k_1^2+1}-1}{k_1}.$$
Make the change of variables
$$\xi = \frac{(x+c)+ty}{\sqrt{1+t^2}} = \cos\gamma\cdot(x+c)+\sin\gamma\cdot y,$$
$$\eta = \frac{-t(x+c)+y}{\sqrt{1+t^2}} = -\sin\gamma\cdot(x+c)+\cos\gamma\cdot y.$$
The inverse change of variables has the form
$$x+c = \frac{\xi - t\eta}{\sqrt{1+t^2}},$$
$$y = \frac{t\xi+\eta}{\sqrt{1+t^2}}.$$
The new coordinate system $\xi$, $\eta$ is orthogonal, its origin $\xi = 0$, $\eta = 0$ coincides with the point $F_1 = (-c, 0)$ (in the $x, y$-coordinates), and the $\xi$-axis (given by the equality $\eta = 0$) is the bisector of the angle $KF_1F_2$ formed by the lines $y = 0$ and $y = k_1(x+c)$.

In the new coordinates $\xi$, $\eta$ the figure $F_{\{x,y\}}$ takes the following form:
$$\begin{aligned} F_{\{\xi,\eta\}} = \Big\{(\xi,\eta) :\ & \\ \frac{(\xi-t\eta)^2}{\alpha^2} - \frac{(t\xi+\eta)^2}{\beta^2} < 1+t^2 < & \frac{(\xi-t\eta)^2}{a^2}+\frac{(t\xi+\eta)^2}{b^2}, \\ & k_1 < \frac{t\xi+\eta}{\xi-t\eta} < k_2, \quad t\xi+\eta>0\Big\}. \end{aligned} \tag{15}$$
Let $\tilde{F}_{\{\xi,\eta\}}$ be symmetric to $F_{\{\xi,\eta\}}$ with respect to the line $\eta = 0$; then the two-dimensional figure $F_{\{\xi,\eta\}} \cup \tilde{F}_{\{\xi,\eta\}}$ is invisible from the origin $F_1$ (see Fig. 2).

Indeed, a light ray emanated from $F_1$ makes the first reflection from the elliptic arc bounding $F_{\{\xi,\eta\}}$. The second reflection is from a point on the flat segment bounding $\tilde{F}_{\{\xi,\eta\}}$, besides the distance from $F_1$ to this point equals $|F_1F_2|$. The condition (13) and the inequalities (9) ensure that this point really belongs to the flat segment.

The three-dimensional figures $G_1$ and $G_2$ invisible from the origin are obtained by rotating the figure $F_{\{\xi,\eta\}} \cup \tilde{F}_{\{\xi,\eta\}}$ with respect to the axis $\eta = 0$ and to the axis $\xi = 0$. In the first case (see Fig. 7) the figure $G_1$ is
$$G_1 = \{(u,v,w) : (u, \sqrt{v^2+w^2}) \in F_{\{\xi,\eta\}}\}; \tag{16}$$
in the second case the figure $G_2$ is
$$G_2 = \{(u,v,w) : (\sqrt{u^2+v^2}, |w|) \in F_{\{\xi,\eta\}}\}. \tag{17}$$

**5.** Summarizing, the construction of an invisible body is as follows. Choose the parameters $c > 0$ and $1 < \varkappa < 2$. Calculate $k_{\min}$ and $k_{\max}$ according to the formulas

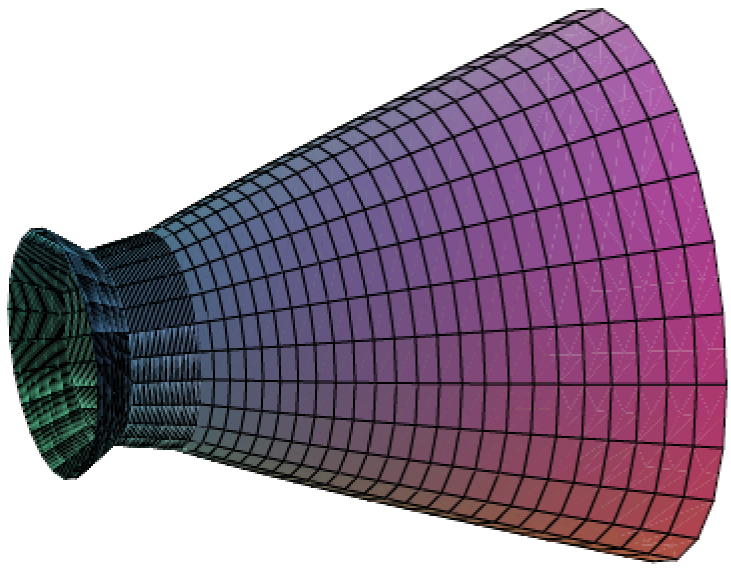

Figure 7: The 3-dimensional body obtained by rotating the plane figure on Fig. 2 around the horizontal axis. In order to make the body's shape more visible, the exterior part of its boundary is removed.

(12) and (8), and choose the parameters $k_1$ and $k_2$ satisfying (13). Define $a^2$, $b^2$, $\alpha^2$, $\beta^2$ by

$$a^2 = \varkappa^2 c^2, \quad b^2 = (\varkappa^2 - 1)c^2, \quad \alpha^2 = \varkappa^{-2} c^2, \quad \beta^2 = (1 - \varkappa^{-2})c^2,$$

and calculate $t$ according to (14). Finally, define the 2D region $F_{\{\xi,\eta\}}$ by (15), and define the regions $G_1$ and $G_2$ in the three-dimensional space of Cartesian coordinates $u, v, w$ by (16) and (17). Each of these regions depends on 4 continuous parameters: scale of the picture $c$, excentricity of the ellipse $\varkappa$, and inclinations of two generating lines, $k_1$ and $k_2$.

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# References

[1] A. Aleksenko and A. Plakhov. *Bodies of zero resistance and bodies invisible in one direction.* Nonlinearity **22**, 2009, 1247-1258.

[2] G. Galperin and A. Plakhov. *A geometric problem that leads to the billiard law of reflection.* Math. Education **15**, 2011, pp.175-180. [Matematicheskoe Prosveschenie, ser. 3, vyp. 15, 2011, 175-180, in Russian.]

[3] A. Plakhov. *Scattering in billiards and problems of Newtonian aerodynamics.* Russian Mathematical Surveys **64**, 2009, 873вЂ“938.

[4] Sergei Tabachnikov *Private communication.*

[5] E. Lakshtanov, B. D. Sleeman, and B. Vainberg. High frequency scattering by a classically invisible body. *SIAM J. Appl. Math.*, 72(2), 2012, pp.646-669.

[6] E. Lakshtanov, and B. Vainberg. A priori estimates for high frequency scattering by obstacles of arbitrary shape *Comm. Part. Diff. Eq.*, 37, 2012, pp.1789-1804.

[7] E. Lakshtanov, and B. D. Sleeman, Acoustically invisible gateways *Inverse Problems and Imaging*, 5:1, 2011, pp.203-217.

[8] A. Plakhov, Exterior billiards. Systems with impacts outside bounded domains, Springer, New York, 2012.

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